Three-Dimensional Filtration on a Circular Leaf

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The unconfined growth of a filter cake on a circular filter cloth was theoretically and experimentally studied as the simplest example of a three-dimensional filtration. The results are compared with the one-dimensional cake growth which ensues when the cake is laterally restrained by a cylindrical sleeve.

Filtration theory to date has emphasized one-dimensional phenomena, in which the surface of the filter cake grows parallel to the filter cloth. Apart from the fact that this is the type most often encountered in applications, the emphasis also stems from the relative ease with which experimental results may be interpreted. Nevertheless it appears advantageous to supplement conventional, one-dimensional leaf tests with more elaborate experiments in which the filter cake is constrained to grow in three dimensions. Phenomena arise which have no counterpart in the former case. These may be experimentally investigated to learn more of the underlying mechanisms governing the basic filtration process.

CIRCULAR TEST LEAF

Experimentally and theoretically the simplest example of a truly three-dimensional filtration occurs when a slurry is filtered (without lateral restraints) on a circular filter cloth, as in Figure 1(a). Attention is confined to the analysis of this case.

The filter cloth behaves as an infinitesimally thin circular disk (radius $= c$) acting as a sink for filtrate passing through the cake. Cake is assumed to build up on only one side of the test leaf. This cake may be achieved experimentally by attaching a thin, flat O-ring of large diameter to the surface of a standard test leaf. Symmetry considerations dictate that the resultant cake be spheroidal. Its equatorial radius at any stage of growth is denoted by a, and its polar radius (thickness) by b. At the outset of the experiment this spheroidal cake takes the form of a thin circular disk having essentially the same radius as the filter cloth. Thicker cakes grow beyond the boundaries of the cloth and approach a hemisphere as their limiting shape.

If cake growth on the test leaf occurred in but one dimension, the resulting cake would adopt the shape of a right circular cylinder, as in Figure 1(b). Its height at any stage of development is denoted by h. It may be realized experimentally by attaching a long cylindrical sleeve of radius c to a standard circular test leaf. This cake is referred to at various places in the text as a one-dimensional, flat, or cylindrical cake, in contrast to the unconfined spheroidal cake. These two cases will ultimately be compared.

As an idealization it is assumed that the filter cakes are incompressible and that the filter cloths offer no resistance to the passage of filtrate through them. In constant pressure filtrations, to which attention is confined, the surface of the cloth is maintained at some constant subatmospheric pressure p_i. The outer cake surface, contacting the slurry, is essentially at atmospheric pressure p_o. Minor differences in hydrostatic pressure are thus neglected in comparison with the pressure drop $\Delta p = p_o - p_i$ across the cake.

OBLATE SPHEROIDAL COORDINATES

The problem of determining the filter cake deposition as a function of time is best discussed in a system of orthogonal curvilinear coordinates of revolution termed oblate spheroidal coordinates (ξ, η, ϕ). A detailed discussion of the properties of this coordinate system is set forth, for example, by Lamb (2) and Magnus and Oberhettinger (3). These coordinates are defined by the relations

$$x = \tilde{\omega} \cos \phi, \ y = \tilde{\omega} \sin \phi, \ z = c\,\xi\,\eta \qquad (1)$$

where

$$\tilde{\omega} = (x^2 + y^2)^{1/2} = c\,\{(1+\xi^2)\,(1-\eta^2)\}^{1/2} \qquad (2)$$

Here (x, y, z) and $(\tilde{\omega}, \phi, z)$ are, respectively, Cartesian coordinates and circular cylindrical coordinates. Each point in space is represented at least once by permitting the spheroidal coordinates to range over the following values:

$$0 \leq \xi < \infty, \ 0 \leq \eta \leq 1, \ 0 \leq \phi < 2\pi \qquad (3)$$

It follows from Equation (1) that the coordinate surfaces $\phi =$ constant are a family of meridian planes containing the z-axis. Moreover elimination of η between the expressions for z and $\tilde{\omega}$ in Equations (1) and (2), respectively, results in the expression

$$\frac{z^2}{c^2\,\xi^2} + \frac{\tilde{\omega}^2}{c^2\,(1+\xi^2)} = 1 \qquad (4)$$

Since $\tilde{\omega}^2 = x^2 + y^2$, it follows that the coordinate surfaces $\xi =$ constant are a confocal family of oblate spheroids having their geometric center at the origin and the z-axis as their axis of revolution. Spheroids of this type are generated by the rotation of an ellipse about its minor axis. The common focal circle of the family lies in the plane $z = 0$ and corresponds to the circle $\tilde{\omega} = c$.

The major and minor semiaxes, a and b respectively, of a typical oblate spheroid $\xi = \xi_o =$ constant, lie in the plane $z = 0$ and along the z-axis, respectively. They are given by the relations

$$a = c(1+\xi_o^2)^{1/2}, \ b = c\,\xi_o \qquad (5)$$

from which it follows that

$$a^2 - b^2 = c^2 \qquad (6)$$

and also

$$\xi_o = b/c \qquad (7)$$

The ellipsoid given by $\xi_o = 0$ is degenerate and corresponds to that portion of the plane $z = 0$ which lies within the focal circle $0 \leq \tilde{\omega} \leq c$.

When ξ is eliminated between Equations (1) and (2), one obtains

$$\frac{-z^2}{c^2\,\eta^2} + \frac{\tilde{\omega}^2}{c^2\,(1-\eta^2)} = 1 \qquad (8)$$

The coordinate surfaces $\eta =$ constant are therefore a family of confocal hyperboloids of revolution of one sheet having the z-axis as their axis of revolution and having the same common focal circle as the family of oblate spheroids. The hyperboloid given by $\eta = 0$ is degenerate and corresponds to that portion of the plane $z = 0$ external to the focal circle; that is $\tilde{\omega} \geq c$.

Figure 2 shows some traces of the coordinate surfaces $\xi =$ constant and $\eta =$ constant in a meridian plane. Also shown are unit vectors $\mathbf{i}_\xi$ and $\mathbf{i}_\eta$ in the ξ and η directions, respectively. The unit vector $\mathbf{i}_\phi$ is directed out of the page.

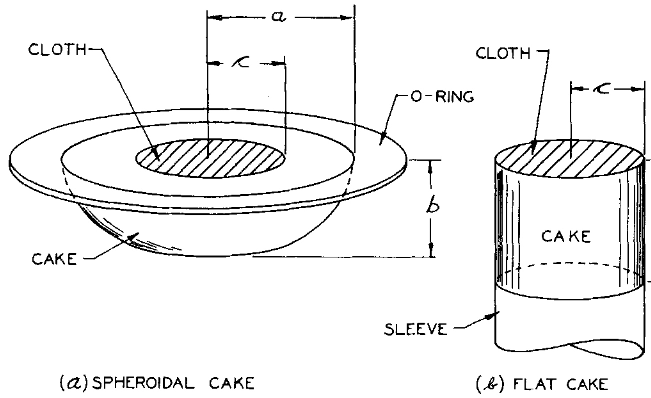

Fig. 1. Filter cake growth on a circular test leaf.

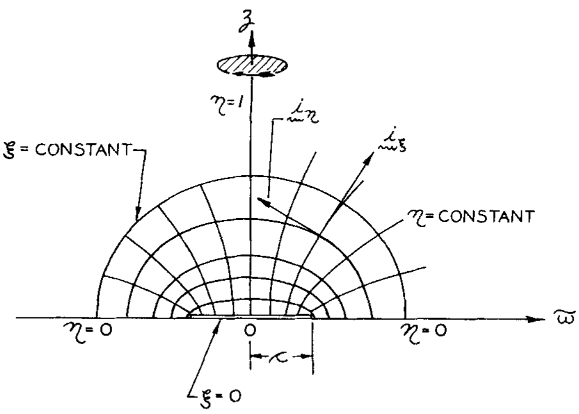

Fig. 2. Oblate spheroidal coordinates in a meridian plane.

The expression for the distance element $ds^2 = dx^2 + dy^2 + dz^2$ in these coordinates is

$$ds^2 = \frac{c^2(\xi^2+\eta^2)}{1+\xi^2}d\xi^2 + \frac{c^2(\xi^2+\eta^2)}{1-\eta^2}d\eta^2 + c^2(1+\xi^2)(1-\eta^2)d\phi^2$$

Upon comparison with the general expression for the incremental distance element in orthogonal curvilinear coordinates

$$ds^2 = \frac{d\xi^2}{h_\xi^2} + \frac{d\eta^2}{h_\eta^2} + \frac{d\phi^2}{h_\phi^2}$$

one obtains the following values for the metrical coefficients

$$h_\xi = \frac{1}{c}\sqrt{\frac{1+\xi^2}{\xi^2+\eta^2}}, \quad h_\eta = \frac{1}{c}\sqrt{\frac{1-\eta^2}{\xi^2+\eta^2}}, \quad h_\phi = \frac{1}{c\sqrt{(1+\xi^2)(1-\eta^2)}} \tag{9}$$

PRESSURE DISTRIBUTION

The instantaneous distribution of filtrate velocities within the cake is governed by Darcy's law

$$\mathbf{v} = -\frac{kg_c}{\mu}\nabla p \tag{10}$$

In addition for an incompressible filtrate

$$\nabla \cdot \mathbf{v} = 0$$

Hence for a homogeneous isotropic filter cake this gives Laplace's equation

$$\nabla^2 p = 0 \tag{11}$$

as the differential equation satisfied by the instantaneous pressure distribution within the cake.

Analysis of the general problem of three-dimensional filtration is complicated by the fact that the instantaneous shape of the filter cake is normally not known á priori. Here however this is not true, since at any time the outer surface of the filter cake will coincide with one of the coordinate surfaces ξ = constant. The validity of this assertion derives from the uniqueness of the physical problem; that is if one regards the statement as a tentative assumption and succeeds in obtaining a solution of the problem which satisfies all the restrictions imposed upon it, then, as the physical problem possesses a unique solution, the assumption is necessarily correct.

The surface of the test leaf coincides with the infinite plane $z = 0$. That part of it, $\tilde{\omega} \leq c$, covered by the circular filter cloth corresponds to the value $\xi = 0$. The remainder of the leaf surface, $\tilde{\omega} > c$, external to the filter cloth corresponds to the value $\eta = 0$. In addition one lets $\xi_o = \xi_o(t)$ define the outer surface of the filter cake at any time t. As $\xi_o = b/c$ is directly proportional to the cake thickness b, it is convenient to refer to this parameter in the text as the dimensionless cake thickness.

The boundary conditions to be satisfied by the pressure are, at the filter cloth surface

$$p = p_i \text{ at } \xi = 0 \text{ for all } t > 0 \tag{12}$$

and at the outer surface of the cake

$$p = p_o \text{ at } \xi = \xi_o \text{ for all } t > 0 \tag{13}$$

The pressures p_i and p_o remain constant for all time. In addition, since the surface $\eta = 0$ is impermeable to filtrate, the normal velocity must vanish there. This requires that

$$v_\eta = 0 \text{ at } \eta = 0 \text{ for all } t > 0 \tag{14}$$

The initial condition is $\xi_o = 0$ at $t = 0$.

In general the Laplacian of p is given by

$$\nabla^2 p = h_\xi\, h_\eta\, h_\phi \left[\frac{\partial}{\partial\xi}\left(\frac{h_\xi}{h_\eta h_\phi}\frac{\partial p}{\partial\xi}\right) + \frac{\partial}{\partial\eta}\left(\frac{h_\eta}{h_\phi h_\xi}\frac{\partial p}{\partial\eta}\right) + \frac{\partial}{\partial\phi}\left(\frac{h_\phi}{h_\xi h_\eta}\frac{\partial p}{\partial\phi}\right)\right]$$

Now the instantaneous pressure in the interior of the cake will be independent of η and ϕ; that is $p = p(\xi; t)$. This leads to the following partial differential equation satisfied by the pressure

$$\frac{\partial}{\partial\xi}\left\{(1+\xi^2)\frac{\partial p}{\partial\xi}\right\} = 0$$

where t is being held constant. Successive integrations yield the solution

$$p = c_1 \tan^{-1}\xi + c_2$$

where c_1 and c_2 may be functions of time. These functions are easily determined from the boundary conditions, Equations (12) and (13). In this way one finds

$$p = p_i + \Delta p\,\frac{\tan^{-1}\xi}{\tan^{-1}\xi_o} \tag{15}$$

where for brevity

$$\Delta p = p_o - p_i \tag{16}$$

denotes the constant pressure drop across the cake. The angle $\tan^{-1}\xi$ is to be measured in radians. Since ξ_o is a function of t, the time variable is implicit in Equation (15).

VELOCITY DISTRIBUTION

Having determined the instantaneous pressure distribution one can now calculate the local velocity in the interior of the cake at any time directly from Darcy's law, Equation (10). The only nonzero velocity component will be v_ξ, the component perpendicular to the surfaces ξ = constant and lying in a meridian plane. This component is given by

$$v_\xi = -\frac{kg_c}{\mu}\,h_\xi\,\frac{\partial p}{\partial\xi}$$

from which one obtains the instantaneous velocity distribution

$$-v_\xi = \frac{k\,\Delta p g_c}{\mu c \tan^{-1}\xi_o}\left[(1+\xi^2)(\xi^2+\eta^2)\right]^{-1/2} \tag{17}$$

That this velocity is negative is merely a consequence of the fact that filtrate is flowing in a direction opposite to that in which ξ is increasing.

Of primary interest is the velocity distribution at the filter cloth surface $\xi = 0$. This is easily found by setting

$\xi = 0$ in the previous equation and observing that $\eta = (c^2 - \tilde{\omega}^2)^{1/2}/c$ at this surface; hence

$$-v_\xi(\xi = 0) = \frac{k \,\Delta p g_c}{\mu \tan^{-1} \xi_o} (c^2 - \tilde{\omega}^2)^{-1/2} \quad (18)$$

The instantaneous volumetric rate $q = q(t)$, at which filtrate leaves the cake (through the cloth), is

$$q = -\int_0^c v_\xi(\xi = 0)\, 2\pi\tilde{\omega}\, d\tilde{\omega}$$

from which one obtains

$$q = \frac{2\pi \,ck\, \Delta p g_c}{\mu \tan^{-1} \xi_o} \quad (19)$$

This flow rate depends on time only through the dependence of ξ_o on time, the other quantities remaining constant throughout the experiment. As cake builds up on the leaf, ξ_o increases and hence q decreases. However unlike one-dimensional filtrations the flow rate does not fall to zero as the cake thickness increases without limit. Rather since

$$\lim_{\xi_o \to \infty} \tan^{-1} \xi_o = \frac{1}{2}\pi$$

the filtrate flow rate and hence rate of cake deposition attain constant limiting values, the former being given by

$$q_\infty = \lim_{t \to \infty} q = \frac{4ck\, \Delta p g_c}{\mu} \quad (20)$$

That a nonzero limiting rate exists stems from the fact that the increasing cake thickness is offset by the increasing mean cross-sectional area of the cake.

The ratio of the instantaneous to ultimate flow rate is thus

$$\frac{q}{q_\infty} = \frac{\frac{1}{2}\pi}{\tan^{-1} \xi_o}$$

For example when the cake thickness is equal to the radius of the leaf ($\xi_o = 1$), the instantaneous filtration rate is twice the final rate.

It also follows from Equation (6) that $b/a \to 1$ as $\xi_o \to \infty$. Thus the *limiting shape of the cake is hemispherical.*

In accordance with Darcy's law, Equation (10), the streamlines for filtrate flow through the cake are orthogonal to the equipressure surfaces ($\xi =$ constant). It follows that the *streamlines lie in meridian planes and there coincide with the curves $\eta =$ constant.* Since the motion is axisymmetric, these streamlines may be represented by Stokes' stream function (2), which here takes the form

$$\Psi = \Psi(\eta; t) = \frac{q}{2\pi}(1 - \eta)$$

The component velocities are related to the above via the relations

$$v_\xi = -\frac{h_\eta}{\tilde{\omega}}\frac{\partial \Psi}{\partial \eta};\; v_\eta = \frac{h_\xi}{\tilde{\omega}}\frac{\partial \Psi}{\partial \xi}$$

In this form the results are analytically identical to the irrotational flow of an ideal fluid through a circular aperture in a plane wall (2).

FILTRATION EQUATION

Let $V = V(t)$ denote the total volume of filtrate collected in time t. Then

$$q = \frac{dV}{dt} = \frac{2\pi ck\Delta p g_c}{\mu \tan^{-1} \xi_o} \quad (21)$$

Now the volume of wet cake on the test leaf at time t (that is when the outer surface corresponds to ξ_o) is given by the volume of the oblate spheroid lying above the plane $z = 0$:

$$Q = \frac{1}{2}[(4/3)\pi a^2 b]$$
$$= (2/3)\pi c^3 \xi_o (1 + \xi_o^2) \quad (22)$$

But the total volume of filtrate collected up to time t is related to the volume of wet cake at this time by a simple mass balance on the solids. They are, in fact, directly proportional to one another, the relationship between them being (*1*)

$$V = \frac{(1 - mw)(1 - \epsilon)\rho_s}{\rho w} Q \quad (23)$$

Eliminating Q between the last two equations one obtains the following relation between filtrate volume and dimensionless cake thickness:

$$V = \frac{2\pi c^3 (1 - mw)(1 - \epsilon)\rho_s}{3\rho w} \xi_o(1 + \xi_o^2) \quad (24)$$

TABLE 1. DRY CAKE WEIGHT RATIO

Dimensionless spheroid thickness, $\xi_o = b/c$	Ratio of dry cake weights, W/W_f
0	1.154
0.2	1.170
0.4	1.219
0.6	1.301
0.8	1.424
1.0	1.577
1.5	2.078
2.0	2.710
3.0	4.260
5.0	8.23
∞	∞

Differentiating with respect to t and setting the resultant expression equal to Equation (21) one gets

$$(\tan^{-1} \xi_o)\, d\{\xi_o(1 + \xi_o^2)\} = \frac{3k\,\Delta p g_c\, \rho w}{c^2 \mu (1 - mw)(1 - \epsilon)\rho_s} dt \quad (25)$$

As is easily verified the left side of this expression is equivalent to

$$d\left[\xi_o(1 + \xi_o^2)\tan^{-1} \xi_o - \frac{1}{2}\xi_o^2\right]$$

Equation (25) may now be integrated subject to the initial condition that there is no cake at zero time; that is $\xi_o = 0$ at $t = 0$. The result is

$$\xi_o(1 + \xi_o^2)\tan^{-1} \xi_o - \frac{1}{2}\xi_o^2 = \left\{\frac{3\,k\Delta p\, g_c\, \rho w}{c^2\mu(1 - mw)(1 - \epsilon)\rho_s}\right\} t \quad (26)$$

Elimination of ξ_o between Equations (24) and (26) now yields the fundamental filtration equation $V = V(t)$. The result is too complex to give explicitly.

The weight of dry, washed solids on the test leaf $W = W(t)$ rather than the filtrate volume V may be the quantity of interest. These are related through the equation

$$W = (1 - \epsilon)\rho_s Q = \frac{V\rho w}{1 - mw} \quad (27)$$

The filtrate volume is no longer proportional to $\sqrt{t}$, as in the one-dimensional case. In fact for long times, that is $\xi_o \to \infty$, one finds from Equations (24) and (26) [or more directly from Equation (20)] that

$$V \to \frac{4ck\Delta p g_c}{\mu} t$$

COMPARISON WITH FLAT CAKE

The growth of the flat, cylindrical cake depicted in Figure 1(b) may be analyzed via the well-known one-dimensional filtration equations (*1*) connecting filtrate volume V_f and filtration time. For the case of negligible cloth resistance and incompressible cakes one has the parabolic formula

$$V_f^2 = \frac{2A^2(1 - mw)\Delta p\, g_c}{\rho w \mu \propto} t$$

where

$$\propto = \frac{1}{k(1 - \epsilon)\rho_s} \quad (28)$$

is the specific cake resistance. In the present instance the filtration area A

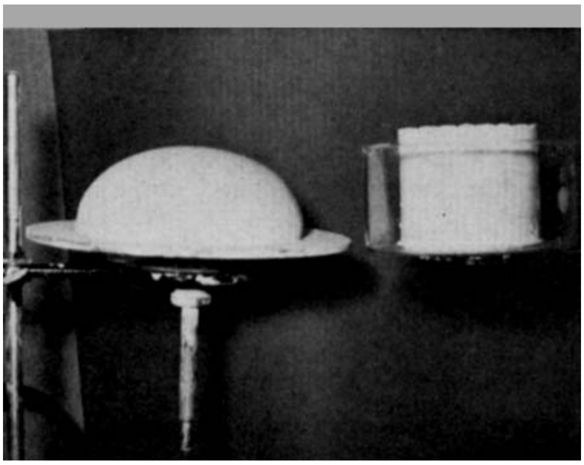

Fig. 3. Photograph of 40-min. filter cakes.

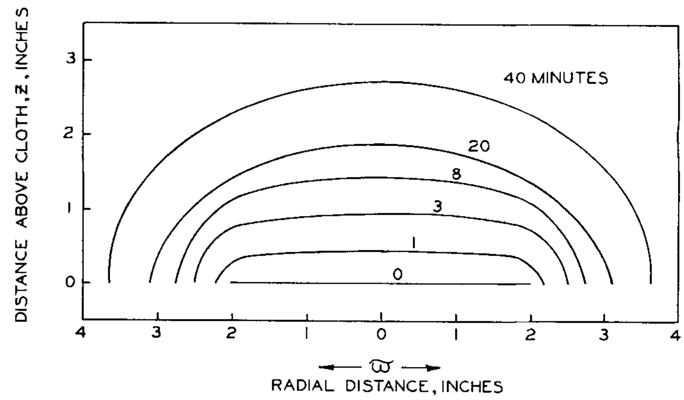

Fig. 4. Experimental cake profiles.

is πc^2. These substitutions result in the expression

$$V_f^2 = \frac{2\pi^2 c^4 \, k \Delta p \, g_c (1 - mw)(1 - \epsilon)\rho_s}{\mu \rho w} t \tag{29}$$

Upon squaring Equation (24) and dividing through by Equation (29) one obtains an expression for the ratio of spheroidal to flat cake filtrate volumes V/V_f in terms of the form time t. As one is interested in comparing these volumes for the same form time, one eliminates the time from the expression obtained above by substituting into it the value of t given by Equation (26). This leads to the comparatively simple expression

$$\frac{W}{W_f} = \frac{V}{V_f} = \sqrt{\frac{2[\xi_o(1 + \xi_o^2)]^2}{3\left[\xi_o(1 + \xi_o^2)\tan^{-1}\xi_o - \frac{1}{2}\xi_o^2\right]}} \tag{30}$$

in which W and W_f are, respectively, the dry weights of the spheroidal and flat cakes.

This formula constitutes a central result of this analysis. It gives the weight ratio of dry spheroidal cake to dry flat cake, for equal filtration times and equal cloth areas, entirely in terms of the dimensionless thickness of the spheroid $\xi_o = b/c$. A few numerical values of this ratio as a function of ξ_o are tabulated in Table 1. In all instances the spheroidal cake is heavier than the corresponding flat cake as would be expected from the enhanced filtration area of the former.

Equation (30) leads to, perhaps, the simplest experimentally testable hypothesis of the theory. Moreover since the relation entails a ratio of quantities, rather than their absolute values, it lessens the severity of the assumptions of negligible cloth resistance and cake compressibility. These effects are similar in both cakes and thereby cancel to some extent in the ratio.

FINITE CLOTH RESISTANCE

At the outset of the experiment ($\xi_o \rightarrow 0$) Equation (30) predicts $W/W_f = \sqrt{4/3} = 1.154$. But this limiting ratio cannot be accepted at face value, for in arriving at it the cloth resistance has been neglected in comparison with that of the cake. Although this assumption may be satisfactory for a sufficiently thick cake, it cannot be correct at the very outset, where the cloth dominates the resistance to flow. Thus in any real situation it should be found that $W/W_f \rightarrow 1$ as $\xi_o \rightarrow 0$.

The present theory therefore provides only the asymptotic values to which the experimental results must conform when the cake attains sufficient resistance for that due to the cloth to become inconsequential by comparison. However unlike one-dimensional filtrations this limiting condition is never truly attained. Rather, since the cake resistance approaches a definite upper limit as its thickness increases, the effect of cloth resistance must persist regardless of cake thickness, eventually reducing the ideal limiting flow rate q_∞ by some definite fraction.

On the assumption that the cloth resistance is equivalent to some uniform, fictitious cake thickness l laid down prior to the experiment, this reduction in limiting rate will depend solely on l/c. A more complete theory therefore requires a family of curves of the form W/W_f = function $(\xi_o, l/c)$. Insight into these phenomena may be gained by considering the closely related but more tractable problem of filtration on a spherical leaf, discussed in the Appendix.

A further consequence of cloth resistance stems from its effect on the shape of the filter cake. The nonuniform velocity distribution [Equation (18)] over that face of the cloth contacting the cake in conjunction with the constancy of the pressure over the other face of the cloth, gives rise to a three-dimensional flow through the cloth. Hence the surface $\xi = 0$ adjoining the cake can no longer be a surface of constant pressure. This implies that the cake cannot maintain its former shape. Now the largest rate of filtrate flow occurs near the cloth edges [$\eta \rightarrow 0$ in Equation (18)]. This flow in turn derives from material deposited along the sides of the cake. Hence this portion of filtrate suffers the greatest resistance to passage through the cloth. This suggests that the effect is to produce a cake which is less flat than the ideal spheroid.

EXPERIMENTS

Experiments were conducted to test some aspects of the theory. Two standard 1/10 sq. ft. circular test leaves were modified as follows. To form the spheroidal cake a thin lucite O-ring, 10 in. in diameter with a 4-in. diameter hole was affixed to one of the leaves. To create the flat filter cake a sleeve comprised of a 3-in. length of lucite tubing, 4 in. in diameter, was attached to a second test leaf.

A #66 cotton cloth was employed. The slurry utilized was a well flocculated 10% (by weight) suspension of U.S.P. grade #300 light precipitated chalk in tap water. True specific gravity of the chalk was 2.71. Mild agitation was provided to prevent settling. The leaves were immersed upside down in the slurry, the depth of cloth submersion being the same for each. Tests were carried out at 29 in. of mercury vacuum. Both experiments were performed simultaneously, that is in parallel, by connecting both leaves to the same vacuum pump through a tee.

A series of six runs was made, their durations being noted in Table 2. No attempt was made to record the instantaneous, cumulative filtrate volumes. At the end of each run the leaves were simultaneously withdrawn from the slurry, turned right-side up, and air allowed to flow through the cakes to maintain their

TABLE 2. EXPERIMENTAL RESULTS

$c = 2$ inches; $\Delta p = 29$ in. Hg.

OBSERVED DATA

Test no.	Time t, min.	Equatorial spheroid radius a, in.	Spheroid thickness b, in.	Flat cake thickness h, in.	Dry cake weight, g. Spheroid, W	Dry cake weight, g. Flat cake, W_f
1	1	2.19	0.43	0.49	57.2	53.4
2	3	2.49	0.93	0.79	105.7	92.5
3*	12*	2.50	1.06	1.38	190.0	156.0
4	8	2.74	1.43	1.50	205.3	159.0
5	20	3.11	1.87	2.28	378.0	266.0
6	40	3.63	2.72	3.31	660.8	362.5

CALCULATED DATA

Test no.	Porosity, ϵ Spheroid	Porosity, ϵ Flat cake
1	0.755	0.805
2	0.829	0.791
3*	—	0.794
4	0.809	0.810
5	0.765	0.791
6	0.801	0.804
Average =	0.792	0.799

* $\Delta P \approx 18$ in. Hg.

shape. A photograph of the cakes obtained during the 40-min. run is shown in Figure 3. The spheroidal cake is still on the test leaf, whereas the cylindrical cake has been removed from its leaf.

Measurements of the dimensions a, b, and h were made for each run. They are recorded in Table 2. Profiles of the spheroids were obtained by forcing a thin piece of sheet metal edgewise across a diameter of the cake. When withdrawn, a thin layer of moist cake adhered to its surface and thereby furnished an imprint of the cake contour. These profiles were subsequently traced onto graph paper. They are shown in Figure 4 for all cakes save the 12-min. one, of which no permanent record was made.

After removal from their leaves by a reverse air blow the cakes were oven-dried and subsequently weighed. Dry cake weights are noted in Table 2.

The (wet) volumes of the spheroidal cakes were obtained by graphical integration of their respective profiles, in accordance with the formula

$$Q = \int_0^b \pi \tilde{\omega}^2 dz$$

which applies to bodies of revolution. Here $\tilde{\omega} = \tilde{\omega}(z)$ is the radius of the spheroid at the elevation z.

The volumes occupied by the cylindrical cakes were obtained directly from the relation $Q_f = \pi c^2 h$. These cake volumes, coupled with the corresponding dry cake weights and the true density of the chalk, enabled the cake porosities to be determined. Calculated porosities are tabulated in Table 2. On the average the porosities of the spheroidal and flat cakes agreed quite closely.

RESULTS

Experimental values of W/W_f vs. $\xi_o = b/c$ are plotted in Figure 5 for each of the six runs. Also shown are the theoretical values. As anticipated the experimental results lie consistently below the theoretical and extrapolate smoothly to the origin. The percent discrepancy between the two decreases monotonically with increasing cake thickness, being only 5% for the thickest cake investigated.

This comparison constitutes a relative test of the spheroid theory, since it also utilizes data for the cylindrical cake. A simple, absolute test of the spheroid theory is furnished by the measured dimensions a and b of the latter. In accordance with Equation (6) these ought to be related by $b = \sqrt{a^2 - c^2}$. Figure 6 is a plot of the ratio of experimental to theoretical cake thickness for the measured value of a. The observed cake tends to be significantly flatter than the ideal cake, although the percent divergence decreases with increasing cake thickness.

Cloth resistance cannot be invoked to explain the discrepancy, as the direction of departure from ideal shape appears opposite to that which may be attributed to it. This flatness may be due to cake compressibility, neglected in the theory. Intuition suggests that the stresses in a compressible medium would result in the observed flatness.

DISCUSSION

The present development provides some insight into the complexity of three-dimensional filtrations. It also suggests ways in which the new phenomena which arise may be advan-

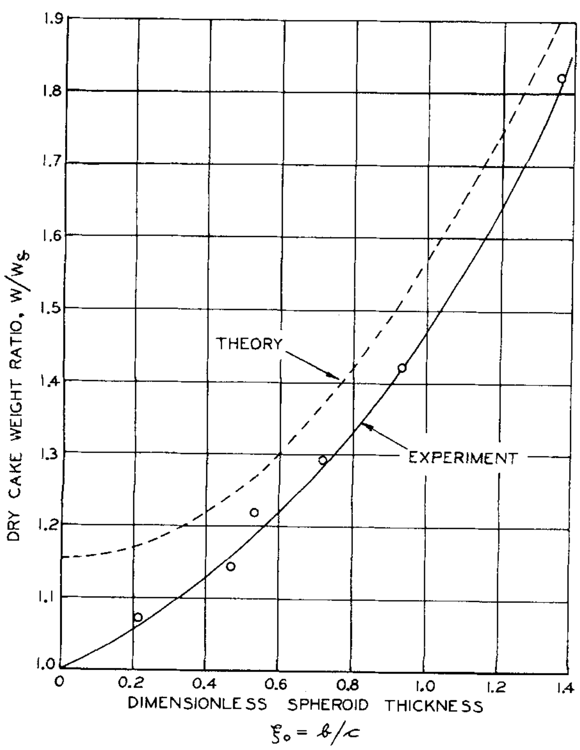

Fig. 5. Experimental vs. theoretical dry cake weight ratios.

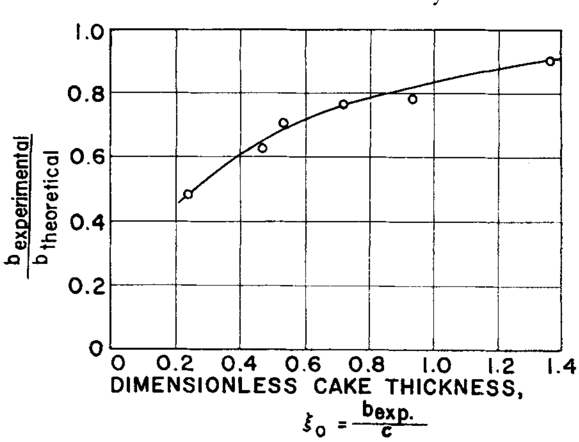

Fig. 6. Ratio of experimental to theoretical cake thicknesses.

tageously employed to gain further understanding of the filtration process. Measurements of the cake profile alone may furnish direct, practically useful information on the compressibility of the cake. No analogous phenomenon exists in the one-dimensional case where the cake surface is invariably parallel to the cloth. Information of this type, even if qualitative, ought to be useful in routine, industrial leaf test work. Three-dimensional experiments have the added advantage of enabling tests to be carried out indefinitely, since the filtration rate does not fall to zero with increasing cake thickness.

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NOTATION

Any consistent system of units may be employed in place of those listed here.

- a = equatorial radius of spheroidal cake, ft.
- A = area of circular filter cloth, sq. ft.
- b = polar radius (thickness) of spheroidal cake, ft.
- c = radius of circular filter cloth, ft.
- g_c = 32.16 (lb.-mass) (ft.)/(lb.-force) (sec.^{2})
- h = height of flat cake, ft.
- h_β = metrical coefficient for β-coordinate, ft.$^{-1}$
- $\mathbf{i}_\beta$ = unit vector in β-direction
- k = filter cake permeability, sq.ft.
- l = thickness of filter cake having same resistance to flow as cloth, ft.
- m = weight ratio of wet cake to dry, washed cake
- p = instantaneous local pressure in cake, lb.-force/sq.ft.
- Δp = $p_o - p_i$ = pressure drop across filter cake plus cloth, lb.-force/sq.ft.
- q = instantaneous volumetric flow rate of filtrate through cake, cu.ft./sec.
- Q = volume of wet cake, cu.ft.
- r = radial distance or radius, ft.
- δs = increment of distance, ft.
- t = time from start of filtration, sec.
- $\mathbf{v}$ = instantaneous local superficial velocity of filtrate in cake, ft./sec.
- v_β = $\mathbf{i}_\beta \cdot \mathbf{v}$ = β-component of above velocity, ft./sec.

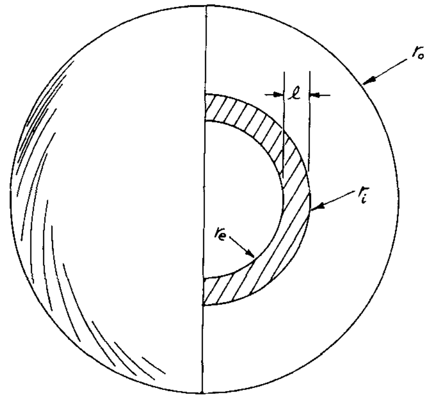

Fig. 7. Filter cake growth on a spherical filtering surface.

- V = total volume of filtrate collected up to time t, cu.ft.
- w = weight fraction of insoluble solids in original slurry
- W = weight of dry, washed filter cake, lb.-mass
- x, y = Cartesian coordinates measured in a plane parallel to filter cloth, ft.
- z = Cartesian coordinate measured normal to filter cloth, ft.

Greek Letters

- α = specific cake resistance, ft./lb.-mass
- ϵ = porosity of filter cake
- η = oblate spheroidal coordinate
- μ = filtrate viscosity, lb.-mass/(ft.) (sec.)
- ξ = oblate spheroidal coordinate
- ξ_o = dimensionless spheroidal cake thickness = b/c
- ρ = density of filtrate, lb.-mass/cu.ft.
- ρ_s = true density of dry, washed solids, lb.-mass/cu.ft.
- ϕ = azimuthal coordinate
- Ψ = stream function, cu.ft./sec.
- $\tilde{\omega}$ = $(x^2 + y^2)^{1/2}$ = cylindrical distance, ft.

Subscripts

- e = equivalent value for the fictitious cake which offers the same resistance as the filter cloth
- f = flat, one-dimensional cake
- i = value at inner surface of filter cake or cloth
- o = value at outer surface of filter cake in contact with slurry
- ∞ = a limiting value at infinite time or for an infinitely thick cake

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APPENDIX

To quantitatively assess the effect of cloth resistance on three-dimensional cake growth consider the simple case of a spherical filter cake forming on a spherical filter cloth of radius r_i, as in Figure 7. Let r denote the distance from the origin to a point in the interior of the cake, and let r_o be the outer radius of the filter cake at any instant of time. The cloth resistance is represented by an equivalent thickness l of cake whose inner radius is r_e and outer radius is r_i. This corresponds to the shaded region in Figure 7.

The boundary conditions to be satisfied are

$$p = p_i \text{ at } r = r_e \text{ for all } t > 0$$

and

$$p = p_o \text{ at } r = r_o \text{ for all } t > 0$$

The pressure distribution satisfying Laplace's equation and the above boundary conditions is

$$p = p(r;t) = p_o - \frac{\Delta p}{\dfrac{1}{r_e} - \dfrac{1}{r_o}} \left(\frac{1}{r} - \frac{1}{r_o} \right)$$

valid for $r \geqslant r_i$. Darcy's law then gives, for the radial velocity at the filter cloth surface, $r = r_i$:

$$-v_r \Big|_{r = r_i} = \frac{k g_c}{\mu} \frac{\partial p}{\partial r} \Big|_{r = r_i}$$

$$= \frac{k \Delta p \, g_c}{\mu \left(\dfrac{1}{r_e} - \dfrac{1}{r_o} \right)} \left(\frac{1}{r_i^2} \right)$$

The instantaneous flow rate through the filter cloth is therefore

$$q = -4\pi r_i^2 v_r|_{r=r_i}$$

$$= \frac{4\pi k \Delta p \, g_c}{\mu \left(\dfrac{1}{r_e} - \dfrac{1}{r_o} \right)}$$

As the cake thickness increases indefinitely, that is $r_o \to \infty$, the limiting flow rate through the cake becomes

$$q_\infty = \frac{4\pi r_e k \Delta p \, g_c}{\mu}$$

If the cloth offered no resistance to filtrate flow, one would have r_i in place of r_e in the above formula. Therefore the limiting rate of cake growth is reduced through the presence of the cloth by the factor

$$1 - \frac{r_e}{r_i} = \frac{l}{r_i}$$

This shows, in three-dimensional filtration, that the effect of cloth resistance persists regardless of cake thickness.